Inverse Scattering in Anisotropic Media

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Abstract. We consider the inverse problem of determining a Riemannian metric in $\mathbf{R}^n$ which is euclidean outside a ball from scattering information. This is a basic inverse scattering problem in anisotropic media. By looking at the wave front set of the scattering operator we are led to consider the "classical" problem of determining a Riemannian metric by measuring the travel times of geodesics passing through the domain. We survey some recent developments on this problem.

1 The inverse scattering problem

Anisotropic materials include most crystals. A common case of aniso-tropy relevant to some Earth structures is transverse isotropy. The inverse scattering problem for this type of media is not very well understood at present. There are many mathematical difficulties associated with the inverse scattering problem for anisotropic Maxwell's equations or the system of elasticity for anisotropic materials. A more basic example of anisotropic media, which involves the study of a scalar partial differential equation, is the case of anisotropic conductors. In this case the electrical conductivity of the medium is represented by a positive definite, symmetric matrix. It is more convenient to look at the conductivity in geometric terms, thus we are going to think of it as a *Riemannian metric*. Notice that this equivalence between Riemannian metrics and conductivities is valid only in dimension $n \geq 3$ [L-U].

Let $g(x) = (g_{ij}(x))$ be a positive definite, symmetric matrix on $\mathbf{R}^n, n \geq 2$. We assume that the Riemannian metric g is smooth (many of the results in this paper are valid assuming finite smoothness). We also assume that the metric is euclidean outside a ball B of radius R centered at the origin, that is $g_{ij} = \delta_{ij}$ for $|x| > R$ where δ_{ij} denotes the Krönecker delta. The euclidean metric is denoted by $e = (\delta_{ij})$. We assume throughout this paper that there are no trapped rays in B, that is any geodesic for the metric g starting at a point in B leaves B in finite time.

We denote by Δ_g the Laplace-Beltrami operator associated to the metric g, i.e. in local coordinates

$$\Delta_g = (\det g)^{-\frac{1}{2}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (\det g)^{\frac{1}{2}} g^{ij} \frac{\partial}{\partial x_j} \tag{1.1}$$

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where $(g^{ij}) = (g_{ij})^{-1}$, $\det g = \det(g_{ij})$. Given $\lambda \in \mathbf{R} - 0, \omega \in S^{n-1}$, the *outgoing eigenfunctions*, $\psi_g(\lambda, x, \omega)$ are solutions of

$$\Delta_g \psi_g + \lambda^2 \psi_g = 0 \tag{1.2}$$

which have the asymptotic behavior

$$\psi_g = e^{i\lambda x\cdot\omega} + \frac{a_g(\lambda,\theta,\omega)e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}} + O(|x|^{-\frac{n-1}{2}-1}) \tag{1.3}$$

where $\theta = \frac{x}{|x|}$. The function $a_g(\lambda, \theta, \omega)$ is called the *scattering amplitude.* It measures, roughly speaking, the amplitude of the radial scattered wave which resulted from the interaction of the incident plane waves $e^{i\lambda x\cdot\omega}$ with the perturbation of the euclidean metric given by g.

The inverse scattering problem is whether one can determine the metric g from a_g, i.e. to study the non-linear map sending g to a_g. It is easy to see that it is not possible to determine the metric uniquely from this information. Let ψ be a smooth diffeomorphism of $\mathbf{R}^n$ which is the identity outside B. We define $v_g = \psi_g \circ \psi^{-1}$. A straightforward calculation shows that v_g satisfies

$$\Delta_{\psi * g} v_g + \lambda^2 v_g = 0 \tag{1.4}$$

where $\psi^* g$ denotes the pull back of the metric g under the diffeomorphism ψ that is

$$\psi^* g = (D\psi \circ g \circ {}^t D\psi) \circ \psi^{-1}.$$

Since the asymptotic behavior of v_g and the ψ_g at infinity is the same we conclude that

$$a_{\psi^* g} = a_g. \tag{1.5}$$

The natural conjecture is that (1.5) is the only obstruction to uniqueness. This conjecture was proven recently. It is a consequence of the paper [B-K] which uses the boundary control method (BC) pioneered by Belishev (see [B] for a survey). In turn this method depends on a Holmgren type uniqueness theorem for hyperbolic equations which was proven by Tataru [T]. See also [R-Z].

The BC method has been greatly extended to solve the inverse scattering problem for any first order and zeroth order selfadjoint perturbation of the Laplace-Beltrami operator [K]. There are also recent results for the case of non-selfadjoint perturbations [K-L].

We remark that if two metrics g_1, g_2 are conformal to each other (i. e. $g_1 = \alpha(x) g_2$ with α a non-zero function) and $\psi * g_1 = g_2$ with ψ a diffeomorphism of $\mathbf{R}^n$ which is the identity outside B then ψ must be the identity and therefore $g_1 = g_2$.

The above mentioned results assume that we know the scattering amplitude for all frequencies and directions. Of course, this is too much information and we would like to measure the scattering amplitude for a more restricted set of angles and frequencies. An interesting physical problem is the *inverse backscattering problem* i.e. we measure $a_g(\lambda, \theta, -\theta)$ for all $\lambda \in \mathbf{R} - 0$ and all $\theta \in S^{n-1}$. The information given depends on n variables. The only known result about this problem is the following: if two metrics are conformal to each other and they are a priori close to the euclidean metric, then the two metrics are the same if their backscattering amplitudes are the same. This is not exactly the result stated in [S-U2] but the methods used there give this result.

Another inverse scattering problem which involves less data is the *fix energy problem*. In this case we measure the scattering amplitude at a fixed frequency λ_0 for all angles $\theta, \omega \in S^{n-1}$. The scattering amplitude $a_g(\lambda_0, \theta, \omega)$ depends on $2n - 2$ variables. It is well known (see for instance [U]) that knowledge of $a_g(\lambda_0, \theta, \omega)$ determines the set of Cauchy data for the Laplace-Beltrami operator on B. Namely we can recover from $a_g(\lambda_0, \theta, \omega)$

$$C_{g,\lambda_0} = \{(u|_{\partial B}, \frac{\partial u}{\partial \nu}|_{\partial \Omega}), \text{ with } u \in H^2(B) \text{ solution of (1.2) on } B.\} \qquad (1.6)$$

Notice that if λ_0 is not a Dirichlet eigenvalue for the Laplace-Beltrami operator then the set of Cauchy data is the graph of the Dirichlet to Neumann map Λ_{g,λ_0}. In the class of metrics conformal to the euclidean metric, it was proven in [Sy-U1] in dimension $n \geq 3$ that C_{g,λ_0} uniquely determines the metric g. In [L-U] it is shown in dimension $n \geq 3$ that C_{g,λ_0} uniquely determines g for real-analytic metrics. The smooth case remains open. The linearization of this problem is studied in [Sy-U2].

In the two dimensional case the anisotropic problem is in some sense easier since we can reduce it to the isotropic case by using isothermal coordinates [A]. In fact in this case the Laplace-Beltrami operator can be transformed, after a change of coordinates, to a conformal multiple of the standard Laplacian. Thus we can transform (1.2) into

$$c^2(x)\Delta + \lambda_0^2$$

with c positive and equal to 1 outside B. In this case it is not known at present for general smooth c whether we can recover c from the scattering amplitude at a non-zero fixed energy. It is known under the a priori assumption that c is small enough [Sy-U3] or for a generic set of $c's$ [Su-U1,2]. The anisotropic conductivity equation, which is the analog of (1.2), is given by

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \gamma^{ij} \frac{\partial \psi_\gamma}{\partial x_j} + \lambda_0^2 \psi_\gamma = 0$$

with $\gamma = (\gamma^{ij})$ a positive definite, symmetric smooth matrix which is the identity outside B. As before the inverse scattering problem at a fixed energy can be reduced to the question of whether the set of Cauchy data C_{γ,λ_0} determines γ uniquely up to conjugation by a group of diffeomorphism which is the identity on the boundary of B. A modification of the method of [A] allows us to reduce the problem to the case of an isotropic conductivity [S]. If $\lambda_0 = 0$ the isotropic problem was solved in [N] (see also [B-U] for another approach that allows for less regular conductivities). For the case $\lambda_0 \neq 0$ uniqueness it is not known at present. Uniqueness has been proven for small enough conductivities or for generic conductivities [Su-U1,2].

In this paper we consider other information obtained from the scattering amplitude which involves less variables than the full scattering amplitude. Namely we will consider the singularities of the scattering operator whose kernel is, essentially, the distribution obtained by taking the Fourier transform of the scattering amplitude in the frequency variable. This leads to the problem of determining a metric from the *scattering relation,* which as we explain in the next section, can be considered as the "classical" analog of the inverse scattering problem. Knowledge of the scattering relation means that if we know the point of entry of the geodesic into B and its direction, we can determine the point of exit of the geodesic from B and the direction of exit. As we also show in section 2 the scattering relation determines, under some additional assumptions, the geodesic distance $d_g(x, y), x, y \in \partial B$ between points in the boundary of the ball. This function measures, roughly speaking, the travel time of geodesics passing through B. The inverse kinematic problem arising in seismology is to determine the metric g from these travel times. We discuss in section 3 this problem in detail. We make emphasis on a new identity which was derived in [S-U2] and played a fundamental role in proving that we can uniquely determine a metric sufficiently close to the euclidean metric (up to isometries) from its travel times. This is formula (3.19). We list in section 4 some open problems.

2 The scattering relation

To define the scattering operator and study its singularities we use the Lax-Phillips of scattering which uses the wave equation to define the scattering operator. It is quite natural in this context to use the wave equation since it is well understood how singularities propagate for solutions of this equation. For more details see [G].

Let $(u_0, u_1) \in C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$. We define the group of operators

$$U_g(t)(u_0, u_1) = (u(t), \frac{\partial u(t)}{\partial t}(t)) \tag{2.1}$$

where u solves the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta_g u = 0 \tag{2.2}$$

We denote by $U_e(t)$ the operator corresponding to the euclidean metric. The $U_g's$ are unitary groups associated to the energy space $\mathcal{H}_g$ defined as the completion of $C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$ under the norm defined by

$$||(u_0, u_1)||_g^2 = \int (|u_0|^2 + \sum_{i,j=1}^{n} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \sqrt{detg})\ dx. \tag{2.3}$$

We denote by $\mathcal{H}_e$ the energy space corresponding to the euclidean metric.

The *Wave Operators* are unitary operators from $\mathcal{H}_g$ to $\mathcal{H}_e$ which are defined by

$$W_\pm = \lim_{t\to\pm\infty} U_e(-t)U_g(t) \tag{2.4}$$

The scattering operator, which is a unitary operator from $\mathcal{H}_e$ to itself, is defined by

$$S_g = W_+ W_-^{-1} \tag{2.5}$$

It follows from finite speed of propagation of solutions of the wave equation that to compute $W_\pm$ acting on compactly supported data we don't need to take the limit in (2.4). Namely for $k \in \mathcal{H}_g$ compactly supported, $W_\pm k = U_e(-t)\circ U_g(t)k$ for $\pm t$ sufficiently large.

We now explain the connection between the scattering amplitude as defined in (1.3) and the scattering operator. In the context of the Lax-Phillips theory of scattering this is seen using a modification of the Radon transform to reduce the problem to a one dimensional problem depending on some parameters.

Let

$$R : \mathcal{E}'(\mathbf{R}^n) \longrightarrow \mathcal{E}'(\mathbf{R}\times S^{n-1})$$

be the Radon transform

$$Rf(s,\theta) = \int_{x\cdot\theta=s} f(x)d\sigma(x) \tag{2.6}$$

where $d\sigma$ is normalized Lebesgue measure on the hyperplane $\{x\cdot\theta = s\}$. Acting in the x-variable, R is defined on those elements of $D'(\mathbf{R}^n\times\mathbf{R}\times S^{n-1})$ having compact support in x for each t,ω.

It is well-known that the Radon transform intertwines the n-dimensional Laplacian with the one-dimensional Laplacian, i.e.,

$$R\Delta u = \frac{\partial^2}{\partial s^2}Ru, u\in\mathcal{E}'(\mathbf{R}^n). \tag{2.7}$$

The modified Lax-Phillips Radon transform [L-P] which maps $\mathbf{C}^2$- to $\mathbf{C}$- valued distributions, is defined by

$$R_{LP}(u_0,u_1) = C_n D_s^{\frac{n-1}{2}}(D_s Ru_0 - Ru_1), \quad n \text{ odd}. \tag{2.8}$$

For n even, in (2.8) one replaces $D_s^{\frac{n-1}{2}}$ by $|D_s|^{\frac{n-1}{2}}$. R_{LP} is a unitary isomorphism from the free energy space $\mathcal{H}_e$ to $L^2(\mathbf{R}\times S^{n-1})$. Furthermore the modified Radon transform has the key property that it intertwines the free group associated to solutions of the wave equation with the translation group on $\mathbf{R}\times S^{n-1}$. Namely we have

$$R_{LP}U_0(u_0,u_1) = T_t R_{LP}(u_0,u_1) \tag{2.9}$$

where T_t denotes the translation group to the right:

$$T_t f(s) = f(s-t), f \in \mathcal{E}'(\mathbf{R} \times S^{n-1}).$$

The scattering operator in "Radon transform land" is defined by

$$\mathcal{S}_g = R_{LP} S_g R_{LP}^{-1}. \tag{2.10}$$

$\mathcal{S}_g$ is a unitary operator from $L^2(\mathbf{R} \times S^{n-1})$ to itself. It is easy to see that it is invariant under translation since the coefficients of the wave equation are independent of t. Thus, $\mathcal{S}_g$ is a convolution operator in the s-variable, which can be written as

$$\mathcal{S}_g f(s,\theta) = If(s,\theta) + \int_{S^{n-1}} k_g(s-s',\theta,\omega) f(\omega) d\omega \tag{2.11}$$

where I denotes the identity operator. The distribution $k_g(s,\theta,\omega)$ is called the *scattering kernel.* We have that

$$a_g(\lambda,\theta,\omega) = c_n \lambda^{n-3} \int_{\mathbf{R}} e^{-is\lambda} k_g(s,\theta,\omega) ds \tag{2.12}$$

where c_n is a constant.

This is a rough outline of the "quantum" picture using the wave equation approach. We describe now the "classical" picture in phase space by computing the singularities of the operators defined above.

It is a well-known result of Hörmander that singularities of solutions of the wave equation propagate along null-bicharacteristics. We consider the principal symbol of the wave equation

$$p(t,x,\tau,\xi) = \tau^2 - h_g(x,\xi) \tag{2.13}$$

with

$$h_g(x,\xi) = \sum_{i,j=1i}^{n} g^{ij} \xi_i \xi_j \tag{2.14}$$

The Hamiltonian vector field associated to p (resp. h_g) is defined by

$$H_p = 2\tau \frac{\partial}{\partial t} + \sum_{j=1}^{n} \frac{\partial p}{\partial \xi_j} \frac{\partial}{\partial x_j} - \sum_{j=1}^{n} \frac{\partial p}{\partial x_j} \frac{\partial}{\xi_j},$$

$$(\text{resp. } H_{h_g} = \sum_{j=1}^{n} \frac{\partial h_g}{\partial \xi_j} \frac{\partial}{\partial x_j} - \sum_{j=1}^{n} \frac{\partial h_g}{\partial x_j} \frac{\partial}{\xi_j}) \tag{2.15}$$

The bicharacteristics are integral curves of the Hamiltonian vector field H_p. The integral curves of H_{h_g} are tangent to the energy surface $h_g = 1$. We denote the bicharacteristic flow of H_p (resp. H_{h_g}) at time t by $\Phi(t)$ (resp. $\Theta_g(t)$.) We remark that the geodesics of the metric g are the projections of the bicharacteristics over x-space. In fact this is another way to define geodesics. $U_g(t)$ is a Fourier integral operator whose canonical relation is given by

$$C_g^{\pm}(t) = \{((x,\xi),(y,\eta)) \in T^*(\mathbf{R}^n) - 0 \times T^*(\mathbf{R}^n) - 0; \quad (t,\tau,x,\xi) = \Phi^t(0,\tilde{\tau},y,\eta), \text{ with } \tilde{\tau} = \pm\sqrt{h_g(y,\eta)}\} \tag{2.16}$$

The "classical" free space is $T^*(\mathbf{R}^n - B) - 0$ together with vector field H_{h_e} The perturbed "classical" space" is $T^*(\mathbf{R}^n) - 0$ together with the vector field H_{h_g}. The natural "classical" analog of the wave operators (2.4) is given by the diffeomorphisms

$$\Psi_\pm = \lim_{t\to\pm\infty} \Theta_e(-t)\Theta_g(t) : T^*(\mathbf{R}^n) - 0 \longrightarrow T^*(\mathbf{R}^n - B) - 0 \tag{2.17}$$

and the "classical" scattering diffeomorphism is given by

$$\Phi_g = \Psi_+ \circ \Psi_-^{-1} : T^*(\mathbf{R}^n - B) - 0 \longrightarrow T^*(\mathbf{R}^n - B) - 0 \tag{2.18}$$

The scattering relation is the graph of Φ_g, that is, for some t

$$\mathcal{R}_g = \{((x,\xi),(y,\eta)) \in (\mathbf{R}^n - B)\times S^{n-1} \times (\mathbf{R}^n - B) \times S^{n-1}; \quad (x,\xi) = \Theta_g(t)(y,\eta)\} \tag{2.19}$$

To know the scattering diffeomorphism Φ_g is equivalent to knowing the scattering relation $\mathcal{R}_g$. Let $\Sigma = \{x\cdot\omega_0 = x_0\cdot\omega_0\}$ be an hyperplane supported in $\mathbf{R}^n - B$ with normal $\omega_0 \in S^{n-1}$ and the point $x_0 \in \mathbf{R}^n - B$ near B.

Under the assumption of no conjugate points on the metric g near B (no caustics) the solution of the Hamilton-Jacobi equation near B

$$H_g(x, d_x S) = 0, \quad S = 0 \text{ on } \Sigma \tag{2.20}$$

is given by $S(x) = d_g(x,\Sigma)$ where $d_g(x,\Sigma)$ denotes the geodesic distance from x to the hyperplane Σ. The Lagrangian manifold Λ obtained by the flow-out from Σ by the integral curves of H_{h_g} tangent to $h_g = 1$ is given by $\Lambda = (x, dS(x))$. To know the scattering relation is equivalent to knowing Λ in $T^*(\mathbf{R}^n - B)$. We then conclude that to know the scattering relation is equivalent to knowing this geodesic distance for all hyperplanes supported outside the ball. Since the metric is euclidean outside B we conclude that to know the scattering relation is equivalent to knowing $d_g(x,y), \forall x,y \in \partial B$. Physically this corresponds to knowing the travel times of geodesics passing through B.

3 The boundary distance function

In the last section we motivated the problem of determining a Riemannian metric on a bounded domain Ω in $\mathbf{R}^n$ from the geodesic distance function $d_g(x,y), x,y \in$

$\partial\Omega$. This function is also called *the hodograph* or the *boundary distance function.* This problem arose in geophysics for the case in which the metric is conformal to the euclidean metric, i.e. $g_{ij} = \frac{1}{c^2}\delta_{ij}$. In the literature this problem is referred to as the *inverse kinematic* problem. The function c models the sound speed of the medium. This problem has also attracted the interest of geometers because of rigidity questions in Riemannian geometry (see for instance [C1,2], [Gr], [M1], [O]). The problem can be formulated for general Riemannian manifolds with boundary.

We now state the problem more precisely. Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary Γ. We assume that $\bar{\Omega}$ is strictly convex with respect to g, i.e., for any two distinct points $x \in \bar{\Omega}$, $y \in \bar{\Omega}$ there is a unique geodesic joining x and y lying entirely in Ω with the possible exception of the endpoints x and y. Let $d_g(x, y)$ denote the geodesic distance between x and y. The inverse problem we address in this section is whether we can determine the Riemannian metric g knowing $d_g(x, y)$ for any $x \in \Gamma$, $y \in \Gamma$. As in (1.5) it is easy to see that g cannot be determined from this information. We have $d_{\psi^* g} = d_g$ for any diffeomorphism $\psi : \bar{\Omega} \to \bar{\Omega}$ that leaves the boundary pointwise fixed, i.e., $\psi|_\Gamma = Id$, where Id denotes the identity map and $\psi^* g$ is the pull-back of the metric g. R. Michel conjectured in [M1] that this is the only obstruction to uniqueness, namely if we have two Riemannian metrics g_1, g_2 with $\bar{\Omega}$ strictly convex with respect to both, and if

$$d_{g_1}(x, y) = d_{g_2}(x, y) \quad \forall (x, y) \in \Gamma^2, \tag{3.1}$$

there exists a diffeomorphism $\psi : \overline{\Omega} \to \overline{\Omega}, \quad \psi|_\Gamma = Id$, so that

$$g_2 = \psi^* g_1. \tag{3.2}$$

As noted earlier in the case that the metrics g_1 and g_2 have the same boundary distance function and are in the same conformal class then the diffeomorphism must be the identity and therefore the conjecture in this case is that the metrics are the same.

In [G-M1,2] (see also [C-S]) the case of radial metric conformal to the euclidean is considered (i.e. the sound speed is assumed to be radial.) The first general result was proven by Mukhometov [Mu], who showed in two dimensions that if two metrics are conformal to the euclidean metric and the domain is geodesically convex for the two metrics then the metrics are the same. Moreover, he proved a stability estimate. The proof is very original and uses a form of an energy inequality for this problem. Energy inequalities are of standard use for hyperbolic equations but Mukhometov's energy inequality was, at the time, completely new. We also note that in two dimensions the problem is formally determined since the hodograph depends on two variables. Mukhometov's result was generalized to higher dimensions in [Mu-R] using a similar method. In [B-G] and [Be] it proved in all dimensions $n \geq 2$, again under the geodesically convex assumption on Ω, that if two metrics are conformal to each other and they have the same hodograph then they must be the same. Also in [B-G], [Be] stability estimates are proved in this case. Croke [C1] gave a nice geometric proof of the uniqueness part of the main result in [B-G], [Be] on Riemannian manifolds with boundary satisfying an additional assumption which is weaker than geodesically convex. As far as we know the stability estimates of [B-G] and [Be] don't follow from Croke's argument.

The conjecture (3.2) has been considered in [Gr], [M1] for general Riemannian manifolds with boundary under some assumptions on the curvature. In particular they have shown that if M is a Riemannian manifold with boundary and Riemannian metric g, geodesically convex with respect to g, the conjecture is valid if M is any compact subdomain of $\mathbf{R}^n$, any compact subdomain of an open n-dimensional hemisphere or any compact subdomain of the hyperbolic plane. In two dimensions in [G-N] the conjecture is proved under some restrictions on the behavior of an extension of the metric to $\mathbf{R}^2$ which are essentially a condition of negative curvature on the extension of the metric. The latter result was generalized in [C2],[O] in two dimensions for negatively curved surfaces with boundary under less stringent condition that geodesically convex and to n-dimensional negatively curved Riemannian manifolds with boundary under additional restrictions [C1].

3.1 The linearized problem

We consider the linearization of the map

$$g \longrightarrow d_g \tag{3.3}$$

in the direction of a $C_0^\infty(\Omega)$-tensor field $f_{ij}, i, j = 1, ..., n$. We consider the Hamiltonian vector field H_{h_g} and we denote by $(x(t), \xi(t))$ the integral curves of H_{h_g} on the energy level $h_g = 1$. We are going to use the following parameterization of those integral curves. Let us denote

$$ST^*\partial\Omega_- := \{(z, \omega) \in ST^*\partial\Omega;\ z \in \partial\Omega,\ \omega \in S^{n-1},\ g^{-1}\omega \cdot \nu(z) \leq 0\}.$$

where $\nu(z)$ is the outer unit normal to $\partial\Omega$. Let us introduce the measure $d\mu(z, \omega) = g^{-1}\omega \cdot \nu(z) dS_z d\omega$ on $ST^*\partial\Omega_-$, where dS_z and $d\omega$ are the surface measures on $\partial\Omega$ and S^{n-1}, respectively. Then $(x(t), \xi(t)) = (x(t; z, \omega), \xi(t; z, \omega))$ is defined as the integral curve of H_{h_g} issued from $(z, \omega) \in ST^*\partial\Omega_-$.

Now we define the geodesic X-ray transform by

$$I_g(f)(z, \theta) = \int_\gamma \sum_{i,j=1}^{3} f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\, dt, \tag{3.4}$$

where $\gamma(t) = \gamma(t; z, \theta)$ is the geodesic issued from, $(z, \theta) \in ST\partial\Omega_-$ parameterized by its arc-length. Here $ST\partial\Omega_-$ consists of all unit (with respect to the metric) vectors on the boundary pointing inside Ω. Since the tangent vector to the geodesic is related to the covector or ξ by the formula $g\dot{\gamma} = \xi$, we get

$$I_g(f)(z, \theta) = \int_\gamma \sum_{i,j=1}^{3} m^{ij}(\gamma(t))\xi_i(t)\xi_j(t)\, dt \tag{3.5}$$

where $m = g^{-1} f g^{-1}$ or, in coordinates $m^{ij} = \sum_{i',j'} g^{ii'} f_{i'j'} g^{j'j}$. It is easy to see that if

$$d_{g+tf} = 0, \forall t$$

then

$$I_g(f) = 0. \tag{3.6}$$

Of course the transform $f \to I_g(f)$ is not injective because the distance function is invariant under change of variables which are the identity at the boundary. For the linearized problem this corresponds to $I_g(dv) = 0$ for any tensor field v satisfying $v|_\Gamma = 0$. Here dv denotes the symmetric covariant derivative of v.

By Theorem 3.3.2 of [Sh] we can uniquely decompose the tensor f_{ij} into its solenoidal and potential parts, i.e.

$$f = f^s + dv, \quad v|_\Gamma = 0 \tag{3.7}$$

The natural conjecture is that

$$I_g(f^s) = 0 \Longrightarrow f^s = 0. \tag{3.8}$$

This conjecture has been proved in [Sh] for Riemannian manifolds with boundary satisfying a positive bound on the sectional curvatures of g. In other words the curvature cannot be too big. We remark that integral geometry of tensor fields has also been extensively studied in the very nice book [Sh]

3.2 The local problem

As mentioned earlier conjecture (3.8) has only been proved under the assumption of constant or negative curvature on the metric g. The case of positive curvature remains open.

The first local result, for metrics sufficiently close to the euclidean metric, was proven in [S-U2]. We now state the result. We denote by $C^k_{(0)}(\Omega)$ the set of all $f \in C^k(\bar{\Omega})$ such that $\partial^\alpha f = 0$ on $\partial\Omega$ for $|\alpha| \leq k$. Then we have.

Theorem 1. *Suppose that g_1 and g_2 are two metrics satisfying (3.1). Then there exists $\varepsilon > 0$, such that if*

$$g_m - e \in C^{12}_{(0)}(\Omega), \quad ||g_m - e||_{C^{12}(\bar{\Omega})} < \varepsilon, \quad m = 1, 2, \tag{3.9}$$

then there exists a C^{11} diffeomorphism $\psi : \bar{\Omega} \to \bar{\Omega}$ such that $\psi|_\Gamma = Id$ and $\psi^* g_1 = g_2$.

The proof of Theorem 1 relies on deriving a new identity (see (3.19)) for the difference of the metrics and working in suitable chosen coordinates. The linearized version of the identity at the euclidean metric gives, roughly speaking, that the integrals along the geodesics (lines in the linear case) of the difference of the two metrics is zero. Then one concludes that the metrics are the same in those coordinates by inverting the X-ray transform. In section 3 of [S-U2] Theorem 3.1 was proven by using the identity and a perturbation argument that leads to the inversion of a Fourier integral operator. It is likely that this identity will also give stability estimates and it will have other applications.

Recently Croke, Daiberkov and Sharafutdinov [C-D-S] have proven a different type of local result. In [C-D-S] it is assumed that the metric g satisfies the same curvature condition under which (3.8) is valid. Now assume that g_1 is a metric sufficiently close to g in an appropriate topology. It is shown in [C-D-S] that the two metrics are isometric. This result doesn't imply Theorem 3.1 since the neighborhood of g depends on the metric g. Eskin [E] wrote an article a few months earlier than the paper [C-D-S] was submitted proving a similar result but assuming that the curvature of g is sufficiently small.

3.3 The main identity

In this section we give complete details of the identity proved in [S-U2].

Assume that we have two metrics g_1 and g_2 satisfying

$$g - e \in C^k_{(0)}(\Omega), \quad \|g - e\|_{C^k(\bar{\Omega})} < \varepsilon \tag{3.10}$$

with some $k \geq 2$ and $\varepsilon > 0$. Assume also that they satisfy (3.9). By (3.10), g_1 and g_2 can be extended outside Ω as e and the so extended metrics belong to $C^k(\mathbf{R}^3)$. From now on we will denote by g_1 and g_2 the extended metrics.

Let $x^{(0)} \in \Gamma, \xi^{(0)} \in S^2$ such that $\nu(x^{(0)}) \cdot g^{-1}\xi^{(0)} < 0$. The integral curves of $H_{h_{g_j}}, j = 1, 2$ tangent to the energy $h_{g_j} = 1$ are denoted by $(x_{g_j}, \xi_{g_j}), j = 1, 2$. They solve the Hamiltonian system

$$\begin{cases} \frac{d}{ds}x_m = \sum_{j=1}^3 g^{mj}\xi_j, & \frac{d}{ds}\xi_m = -\frac{1}{2}\sum_{i,j=1}^3 \frac{\partial g^{ij}}{\partial x_m}\xi_i\xi_j, \quad m = 1, 2, 3, \\ x|_{s=0} = x^{(0)}, & \xi|_{s=0} = \xi^{(0)}. \end{cases} \tag{3.11}$$

Here g is either g_1 or g_2, while the initial conditions are the same for both metrics. We remark that if $\xi^{(0)} \cdot g^{-1}\xi^{(0)} = 1$, then s is the arc-length in (3.11). The assumption (3.1) implies the following property.

Lemma 1. *Let g_1, g_2 be two Riemannian metrics in $\bar{\Omega}$ with $\bar{\Omega}$ strictly convex with respect to anyone of them and assume $g_1|_\Gamma = g_2|_\Gamma$. Assume also (3.1). Let x_{g_m}, ξ_{g_m}, $m = 1, 2$, be the solution of (3.11) with the same initial conditions*

$$x_{g_1}(0) = x_{g_2}(0) = x^{(0)}, \quad \xi_{g_1}(0) = \xi_{g_2}(0) = \xi^{(0)}.$$

Then

$$x_{g_1}(t) = x_{g_2}(t) \in \Gamma, \quad \xi_{g_1}(t) = \xi_{g_2}(t), \tag{3.12}$$

where t is the common length of the corresponding geodesics joining $x^{(0)}$ and $x_{g_1}(t) = x_{g_2}(t)$ provided that $\xi^{(0)} \cdot g^{-1}\xi^{(0)} = 1$.

Proof. This is a direct consequence of the discussion in section 2. Namely if the distance function for two metrics is the same then their scattering relation is the same. We give another proof due to Michel [M1]. Let x_{g_1} be the geodesics related to g_1 defined above. Denote by $s \mapsto y_{g_2}(s)$ the geodesics associated to g_2 joining $x_{g_1}(0)$ and $x_{g_1}(t) \in \Gamma$, where t is the length of $x_{g_1}x_{g_2}$. In other words, $y_{g_2}(0) = x_{g_1}(0)$, $y_{g_2}(t) = x_{g_1}(t)$. Note, that t is also the length of that geodesic. By [M1, Corollary 2.3], the geodesics x_{g_1} and y_{g_2} are tangent at the common endpoints. Since y_{g_2} solves (3.11) with $g = g_2$ and initial data $y_{g_2} = x^{(0)}$, $\xi(0) = \eta^{(0)}$ with some $\eta^{(0)}$, we get that $\eta^{(0)} = \xi^{(0)}$, because the two metrics coincide on the boundary. Therefore, y_{g_2} solves (3.11) with $g = g_2$ and by the uniqueness of that solution we get that $y_{g_2} = x_{g_2}$. This proves the lemma. □

Consider the Hamiltonian system (3.11) with the following initial conditions

$$\begin{cases} \frac{d}{ds}x_m = \sum_{j=1}^3 g^{mj}\xi_j, & \frac{d}{ds}\xi_m = -\frac{1}{2}\sum_{i,j=1}^3 \frac{\partial g^{ij}}{\partial x_m}\xi_i\xi_j, \quad m = 1,2,3, \\ x|_{s=-\rho} = (-\rho, z), & \xi|_{s=-\rho} = (1,0,0). \end{cases} \tag{3.13}$$

Here $z \in \mathbf{R}^2$, $\rho > 0$ is such that $g = e$ for $|x| > \rho$ and the solution $x = x(s,z)$, $\xi = \xi(s,z)$ depends on the parameter z. If $g = e$, then $x = (s,z) = (s, z_1, z_2)$.

We now introduce as new coordinates $y = (s,z)$. Since the metrics are close to the euclidean metric it is easy to see that the map $\Omega \ni x \mapsto y$ is close to *Id* in the C^{k-1} topology for small $\varepsilon > 0$ and therefore is a diffeomorphism. In the new coordinates $g^{-1} = (g^{ij})$ will have the form

$$(g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g^{22} & g^{23} \\ 0 & g^{23} & g^{33} \end{pmatrix}. \tag{3.14}$$

Notice that g would have a similar form, too.

Denote by ψ_1, ψ_2 the maps $x \mapsto y$ related to g_1, g_2, respectively. Instead of g_1, g_2, consider $\tilde{g}_1 = \psi_1^* g_1$ and $\tilde{g}_2 = \psi_2^* g_2$, respectively. It is easy to see that s is the length parameter in (3.13) and therefore (3.1) implies $\psi_1(\Gamma) = \psi_2(\Gamma)$. So, both ψ_1 and ψ_2 map Ω to a new domain $\tilde{\Omega}$. We also have that $\psi_1 = \psi_2$ outside Ω. Therefore, (3.1) remains true for $\tilde{g}_1$, $\tilde{g}_2$ in $\tilde{\Omega}$ and instead of (3.10) we have

$$\tilde{g}_1 - \tilde{g}_2 \in C^{k-2}_{(0)}(\tilde{\Omega}), \quad \|\tilde{g}_m - e\|_{C^{k-2}(\bar{\Omega})} < C\varepsilon, \quad m = 1,2 \tag{3.15}$$

with some $C > 0$. We aim to prove that $\tilde{g}_1 = \tilde{g}_2$. This would prove Theorem 3.1 , because it would imply $\psi^* g_1 = g_2$ where $\psi := \psi_2^{-1}\psi_1$ would be a diffeomorphism in Ω fixing the boundary. For the sake of simplicity of notation, let us denote the new metrics again by g_1, g_2 and $\tilde{\Omega}$ by Ω.

Denote the solution of (3.11) by $x = x(s, x^{(0)}, \xi^{(0)})$, $\xi = \xi(s, x^{(0)}, \xi^{(0)})$. Let us introduce new notation

$$X := (x, \xi).$$

The solution to (3.11) related to g_1 and g_2, respectively, can therefore be written down as $X_{g_j} = X_{g_j}(s, X^{(0)}) = X_{g_j}(s, x^{(0)}, \xi^{(0)})$.

Set $F(s) := X_{g_2}(t-s, X_{g_1}(s, X^{(0)}))$. Here $t = t(X^{(0)})$ is the length of the geodesics issued from $X^{(0)}$ with endpoint on Γ and t is independent of $g = g_1$ or $g = g_2$. Notice that the x-component of $F(s)$ may not be in Ω but belongs to a neighborhood of Γ small with ε. By (3.12), $F(0) = X_{g_2}(t, X^{(0)}) = X_{g_1}(t, X^{(0)}) = F(t)$. Thus

$$\int_0^t F'(s)\, ds = 0. \tag{3.16}$$

Denote $V_{g_j} := (\partial H_{g_j}/\partial\xi, -\partial H_{g_j}/\partial x)$, $j = 1, 2$. Then

$$F'(s) = -V_{g_2}(X_{g_2}(t-s, X_{g_1}(s, X^{(0)}))) + \frac{\partial X_{g_2}}{\partial X^{(0)}}(t-s, X_{g_1}(s, X^{(0)})) \\ V_{g_1}(X_{g_1}(s, X^{(0)})). \tag{3.17}$$

We claim that

$$V_{g_2}(X_{g_2}(t-s, X_{g_1}(s, X^{(0)}))) = \frac{\partial X_{g_2}}{\partial X^{(0)}}(t-s, X_{g_1}(s, X^{(0)})) \\ V_{g_2}(X_{g_1}(s, X^{(0)})). \tag{3.18}$$

Indeed, (3.18) follows from

$$0 = \left.\frac{d}{ds}\right|_{s=0} X(T-s, X(s, X^{(0)})) = -V(X(T, X^{(0)})) \\ + \frac{\partial X}{\partial X^{(0)}}(T, X^{(0)})V(X^{(0)}), \quad \forall T \tag{3.19}$$

after setting $T = t - s$. Therefore, (3.16), (3.17) and (3.18) combined together imply

$$\int_0^t \frac{\partial X_{g_2}}{\partial X^{(0)}}(t-s, X_{g_1}(s, X^{(0)}))\,(V_{g_1} - V_{g_2})\,(X_{g_1}(s, X^{(0)}))\, ds = 0. \tag{3.20}$$

Formula (3.20) is the main result used in [S-U2] to prove that the metrics coincide. This identity is a non-linear integral equation on the difference of the metrics g_1 and g_2. We formally linearize this identity at the euclidean metric to explain how to prove that the metrics coincide. In other words, we will formally replace X_{g_1} and X_{g_2} by X_e, where e is the euclidean metric, but we will keep V_{g_1} and V_{g_2}.

Suppose $g = e$. Then $X_e = (x_e, \xi_e)$ solves $x_e' = \xi_e$, $\xi_e' = 0$, therefore $V_e = (\xi, 0)$. It is easy to see that in this case

$$X_e = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} X^{(0)}, \quad \frac{\partial X_e}{\partial X^{(0)}} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}. \tag{3.21}$$

Since $V = (g^{-1}\xi, -\frac{1}{2}\nabla_x(g^{-1}\xi)\cdot\xi)$ (recall that $g^{-1} = \{g^{ij}\}$), we get the following formal linearization formula for (3.20)

$$\int_0^t \left(m\xi - \frac{1}{2}(t-s)\nabla_x(m\xi)\cdot\xi,\ -\frac{1}{2}\nabla_x(m\xi)\cdot\xi\right)(x^{(0)} + s\xi)\, ds = 0, \tag{3.22}$$

where $\{m_{ij}\} := \{g_1^{ij}\} - \{g_2^{ij}\}$, $x^{(0)} \in \Gamma$, $\xi = \xi^{(0)} \in S^2$ and $\xi^{(0)} \cdot \nu(x^{(0)}) < 0$. By (3.14), m has the form

$$m = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{23} & m_{33} \end{pmatrix}. \tag{3.23}$$

Equating the second components of both sides in (3.22), we get

$$\int_0^t \sum_{i,j=2}^3 \nabla_x m_{ij}(x^{(0)} + s\xi)\xi_i\xi_j \, ds = 0 \tag{3.24}$$

for $x^{(0)}$ and ξ as above. This equation easily implies

$$\sum_{i,j=2}^3 \eta \hat{m}_{ij}(\eta)\xi_i\xi_j = 0 \quad \text{for } \xi \cdot \eta = 0, \tag{3.25}$$

where $\hat{m}(\eta)$ is the Fourier transform of $m(x)$ extended as 0 outside Ω. Let $p = (0, p_2, p_3) \in S^2$ be a parameter. Picking

$$\xi = \xi_p(\eta) = \frac{\eta \times p}{|\eta \times p|} = \frac{(p_3\eta_2 - p_2\eta_3, -p_3\eta_1, p_2\eta_1)}{\sqrt{\eta_1^2 + (p_3\eta_2 - p_2\eta_3)^2}}, \tag{3.26}$$

we get

$$\eta \frac{p_2^2\eta_1^2\hat{m}_{33}(\eta) + p_3^2\eta_1^2\hat{m}_{22}(\eta) - 2p_2p_3\eta_1^2\hat{m}_{23}(\eta)}{\eta_1^2 + (p_3\eta_2 - p_2\eta_3)^2} = 0. \tag{3.27}$$

Choosing $p = (0, 1, 0)$ yields

$$\eta \frac{\eta_1^2}{\eta_1^2 + \eta_3^2} \hat{m}_{33}(\eta) = 0, \tag{3.28}$$

therefore $m_{33} = 0$. Next, setting $p = (0, 0, 1)$ in (3.27) leads to

$$\eta \frac{\eta_1^2}{\eta_1^2 + \eta_2^2} \hat{m}_{22}(\eta) = 0, \tag{3.29}$$

so $m_{22} = 0$. And finally, choosing $p = (0, 1, 1)/\sqrt{2}$, we obtain

$$\eta \frac{\eta_1^2\hat{m}_{33}(\eta) + \eta_1^2\hat{m}_{22}(\eta) - 2\eta_1^2\hat{m}_{23}(\eta)}{\eta_1^2 + (\eta_3 - \eta_2)^2/2} = 0, \tag{3.30}$$

thus $m_{23} = 0$.

4 Open problems

In this section we mention some open problems directly related to the conjecture (3.2).

- *Boundary determination.* Suppose we know d_g. Can we recover, in appropriate coordinates, all the derivatives of g at the boundary? This result was proven in the two dimensional case in [M2]. If the answer is affirmative it is likely that one can prove conjecture (3.2) for real-analytic metrics. Also, we wouldn't need to assume that the metrics coincide at the boundary in the statement of Theorem 3.1.
- *Compactness* Moding-out by the group of diffeomorphisms which are the identity on the boundary, is the set of metrics having the same boundary distance function compact in some appropriate topology? A result of this kind combined with the local results [C-D-S], [S-U2] would probably lead to a proof that, under appropriate restrictions on the curvature, there is only a finite number of metrics (up to isometry) with the same boundary distance function.
- *The two dimensional case* In this case we can use isothermal coordinates [A] to reduce the problem to the isotropic case. The problem is that the change of variables produced in this fashion is not the identity at the boundary and we cannot use Mukhometov's result. It is easy to see that it is enough to prove that the change of variables resulting at the boundary is the boundary value of a conformal map.
- *Caustics* Most of the results mentioned in this paper on the conjecture assume that the domain (or manifold) is geodesically convex. It is very easy to find counterexamples if the function d_g is multivalued [G-M1]. However, the scattering relation is well defined by just assuming that there are no trapped geodesics. Is it possible to generalize the known results about recovering the metric from the boundary distance function to recover the metric (up to isometry) from the scattering relation?
- *The Dirichlet to Neumann Map* It was proven in [Sy-U2] that from the hyperbolic Dirichlet to Neumann map we can recover the boundary distance function, assuming again that Ω is geodesically convex. Is there any connection between the elliptic Dirichlet to Neumann and the boundary distance function d_g? As mentioned above to know the elliptic DN map is the same as knowing the set of Cauchy data (1.6). This set is vaguely resemblant of the scattering relation (2.16).

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